INFORMATION, LEARNING, AND THE STUDY OF REPEATED CHOICES BY INDIVIDUALS*

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I provide a model of learning and repeated choices over a finite set of discrete alternatives that is consistent with random utility maximization, and derive the probability that an individual will move between the alternatives. The model provides a framework for studying and testing the effects of information arrival and learning on sequences of choice decisions. There are three key features of this new model. First, as the time between choices becomes small, the model has a special case a standard cross-sectional model of discrete choice. Second, the model provides a framework that allows for choices to occur at arbitrary intervals of time, rather than requiring that all choices occur at the same intervals as is true in traditional panel survey designs. And third, the model alleviates restrictions imposed by other models on how the systematic component of utilities may change over time. I relate these new results to existing methods of estimating transition probabilities.

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1 Introduction

Individuals often face the same choices again and again. For example, an individual may shop for the same type of good and each time faces a choice among competing brands. Or a citizen may be repeatedly interviewed as part of a panel of surveys and asked each time to give their current vote intention. These are the types of repeated choices that I analyze in this paper: choices of fixed, mutually exclusive alternatives that are made repeatedly at varying intervals of time.

I propose a theory of information arrival and learning by individuals that I incorporate into a model of individual choice behavior. The resulting dynamic model of choice for discrete alternatives is consistent with random utility maximization, and has standard cross-sectional models of choice behavior as a special case. The model is characterized by each individual combining their personal preferences or private information with systematic and public features of the choices that are beyond the control of the individual.

There are two key innovations of this model. First, the decoupling of private preferences and the public/systematic information about the utility of the alternatives allows for exogenous change in the systematic utility without the previous systematic value necessarily having an enduring impact which affects subsequent choices. For example, the price of a good may be altered without affecting the impact the idiosyncratic, non-price related preferences of a consumer. Second, as the time between choices shrinks, the probability of making the same choice increases, and as the time between choices grows the choice probabilities become independent of the previous choice.

This paper proceeds as follows. In Section 2 I offer a brief review of the historical context of the current research. In Section 3, I propose a theory of intertemporal learning and choice behavior, and also present previous theories. Section 4 provides the main result of the paper, in particular I derive the transition probabilities for my theory of intertemporal choice. In Section 5, I provide examples and simulations to illustrate the theoretical concepts developed in the previous section.

2 Review

The class of models that are considered in this paper are premised on individuals keeping track of a stochastic process, and choosing the discrete alternative that provides the greatest utility at the time of the choice being made. Like many choice models, the stochastic component of the model follows the Type I extreme value distribution,[1] which is in the domain of attraction for the maxima of a large class of distributions (Gnedenko, 1943; Resnick, 1987).

Following the axiomatic foundations of choice behavior provided by Luce (1959), extreme value distributions have had a central and in general unique role in the statistical analysis of choice behavior, in particular the double exponential or Type I extreme value distribution (Luce and Suppes, 1965; McFadden, 1974; Yellot, 1977). The derivation of the multinomial logit model of choice standardly begins with positing a Type I extreme value distribution for the errors associated with each choice (Luce and Suppes, 1965; McFadden, 1974; Amemyia, 1985). McFadden (1974) showed that this distribution is uniquely consistent with the axioms of choice proposed by Luce.

Dagsvik (1983, 1988) subsequently derived a dynamic choice model which has as special cases the cross-sectional models of choice commonly analyzed (logit, GEV, etc.). Resnick and Roy (1990) provide a parsimonious framework for analytically characterizing repeated choices, and avoid the combinatorics of Dagsvik's approach by taking advantage of the properties of compound stochastic processes (Dwass, 1964; Resnick, 1987). The existing theories of intertemporal choice behavior however restrict how the differences between mean utilities of choices may change over time. Specifically, systematic utility cannot in general decrease.[2] I provide analytical results within a framework where individuals keep track of information about idiosyncratic and systematic utility separately for each choice, so it is possible that the total utility for a choice can be reduced by a decrease in the systematic utility while the idiosyncratic utility remains unchanged. For a cross-sectional analysis, both the new and prior models yield the same analytical results for choice probabilities.

Two examples illustrate the importance of having a model of dynamic choice that does not place restriction on the systematic components of utility,

Example 1: In spatial models of voting, utility from a candidate is affected by the distance of the candidate's policy positions from an individual's ideal point. Individuals may also have idiosyncratic (unobserved) disposition toward the candidate which are randomly distributed across the population. If a candidate moves further away from the ideal point of a voter, decreasing the systematic utility of that candidate, the

[1] Also referred to simply as *the* extreme value distribution. Named references to this same distribution include the "log Weibull" and "Gumbel" distribution (Casella and Berger, 1990), "Gnedenko" and "Weibull" distribution (McFadden, 1974).

[2] This restriction can be slightly weakened to allow for means which are deterministic smooth functions of time (Dagsvik, 2002).

voter should be less inclined to support the candidate on average. As soon as this change in policy position is observed, the probability of supporting the candidate should change. If a candidate's policy moves away from an individual's ideal point, the fact that the candidate was once closer to the candidate should not in general prevent the probability of supporting the candidate from declining.

Example 2: In models of consumer purchasing behavior, consumers may base their choice on observable factors such as price, but may also be swayed by unobserved perceptions of different brands. After an increase in price the mere fact that the price for the good had previously been low should not in itself mean that the cumulative utility for the good remains high, except potentially through other processes such as state dependence of those who had previously been induced to try the product.

3 Theories of Information Arrival and Learning

In this section I consider models of information and individual behavior. Each individual is repeatedly face a choice between mutually exclusive alternatives indexed by $\mathcal{J} = \{1, 2, \ldots, K\}$, and must select a single option when making a choice. As time passes, each individual receives a stream of private information about the personal utility of each alternative which, in combination with the objective and commonly known information about the choices, determines the choice of each individual. The individual chooses the option which offers the greatest combined personal and publicly known utility.

Specifically, the beginning of the arrival of information is denoted $t_{i0} = 0$ and an interval of time between the sth choice decision by person i and her prior choice decision is denoted $(t_{i,s-1}, t_{is}]$. During each interval of time, choice alternative j is associated with a sequence of information observed by individual i. Some of this information is a sequence of publicly observable (or estimable) values $\mu_{ij}(0,t]$, while other information $v_{ij}(0,t]$ is personal, idiosyncratic, and known only to the individual. Although the baseline utility of an alternative is assumed to be observable (or estimable) by others, it may vary systematically across individuals and therefore may be indexed by i. Since the literature uses various notations, let the values of processes which span the entire time up to time t be defined using the following equivalently notation to represent the J-vectors $\mu_{it} = \mu_i(0,t] = \mu_i(t)$ and $v_{it} = v_i(0,t] = v_i(t)$. Note, all operations comparing vectors are performed element-wise, and $\bigvee$ is the maximum function. Analytical forms will be derived in terms of a single individual, and therefore the i subscript will be omitted.

The intertemporal theory I propose treats the observations as snapshots of an evolving learning process. Individuals do not discard past private signals about the utility of the alternatives, or draw on completely new and independent information each time they make a decision. Each person combines public/commonly believed facts about the alternatives (e.g., prices) with her own idiosyncratic preferences or private information to make the choice. The model I derive assumes that judgments about the baseline and private information evolve separately, and that individuals apply a cumulative extremal process only to their own private information.

These assumptions are summarized as Assumption UP1. The consequence of decoupling the baseline utility from the cumulative extremal process enables researchers to consider situations where baseline utilities may vary arbitrarily over time.

Previously proposed assumptions about the utility process for models of intertemporal choice (Dagsvik, 1983, 1988, 2002; Resnick and Roy, 1990), is that the vector of utilities for the choices is a cumulative extremal process of an additively separable utility function, This assumption is summarized as Assumption UP2. The need for restrictions on The definition of the utility process implies a restriction on how the systematic utility μ may vary over time. To illustrate the restriction, consider a binary choice set, $J \in \{1,2\}$, such that $J(t) = 1$ if

$$\bigvee_{0<t_j\leq t} \mu_t(1) + v_{t_j}(1) > \bigvee_{0<t_j\leq t} \mu_t(2) + v_{t_j}(2).$$

Let the systematic mean of category one decrease at time $t+\tau$, $\mu_t(1) > \mu_{t+\tau}(1)$. For a given $\mu_{t+\tau}(1)$, $-\infty \leq \mu_{t+\tau}(1) < \mu_t(1)$, there exists a $\tau > 0$, for which the cumulative maxima remains unchanged $\mathbf{U}'(t+\tau) = \mathbf{U}'(t)$, and therefore choice $J(t+\tau) = J(t) = 1$. If all else remains equal, except that the systematic component of $\mu_{1,t+\tau}$ becomes arbitrarily undesirable for all individual's, the probability of still choosing category one $P(J(t+\tau)) = 1$ for some $\tau > 0$ because $\mathbf{U}'(t+\tau) = \mathbf{U}'(t) \bigvee \mathbf{U}'(t, t+\tau] = \mathbf{U}'(t)$.

It is useful to also contrast the assumptions underlying a pooled cross-sections, assuming independent and non-cumulative stochastic innovations in each period. These assumptions are summarized in Assumption UP3. The difference between assumptions of UP1 and UP3 lies in the serial correlation between choices over time induced by the cumulative extremal process of ε in UP1. In pooled cross-section models, the vector stochastic utility is assumed to be drawn anew and independently of the last draw of stochastic each time

ASSUMPTION UP1 ("Personal preferences" Utility Process)

$$\mathbf{U}(t,t+\tau] = \mu_{t+\tau} + \varepsilon(t,t+\tau] \quad where \quad \varepsilon(t,t+\tau] = \bigvee_{t<t_j\leq t+\tau} \mathbf{v}_{t_j}$$

(i) *Independence :* $\mathbf{U}(0,t] \perp \mathbf{U}(t,t+\tau]$ *and* $\mathbf{v}(0,t] \perp \mathbf{v}(t,t+\tau]$

(ii) *Continuity:* $\mathbf{U}_t$ *is a continuous random variable*

(iii) *Baseline utilities may vary arbitrarily*

ASSUMPTION UP2 ("Best-Ever" Utility Process)

$$\mathbf{U}'(t,t+\tau] = \bigvee_{t<t_j\leq t+\tau} \mu_{t_j} + \mathbf{v}_{t_j}$$

(i) *Independence :* $\mathbf{U}'(0,t] \perp \mathbf{U}'(t,t+\tau]$ *and* $\mathbf{v}(0,t] \perp \mathbf{v}(t,t+\tau]$

(ii) *Continuity:* $\mathbf{U}'_t$ *is a continuous random variable*

(iii) *Non-decreasing baseline utilities:* $\mu_t < \mu_{t+\tau}$

ASSUMPTION UP3 ("Amnesia" Utility Process)

$$\mathbf{U}^*(t,t+\tau] = \mu_{t+\tau} + \nu(t,t+\tau] \quad where \quad \nu(t,t+\tau] = \bigvee_{t<t_j\leq t+\tau} \nu_{t_j}$$

(i) Independence : $\mathbf{U}^*(0,t] \perp \mathbf{U}^*(t,t+\tau]$ and $\mathbf{v}(0,t] \perp \mathbf{v}(t,t+\tau]$

(ii) Continuity: $\mathbf{U}^*_t$ is a continuous random variable

(iii) Baseline utilities may vary arbitrarily

ASSUMPTION SP (Private Information Stochastic Process) *Private information for a choice,* $v(t,t+\tau]$ *arrives according to a stochastic process, with properties*

(i) *cumulative distribution function* $\Lambda(v_i(t,t+\tau]) = e^{-\tau e^{-v_i}}$ *(Type I extreme value)*

(ii) *Independence:* $E(v_{it} \mid \mu_{it}) = E(v_{it})$

ASSUMPTION CP (Choice process) *An individual's choice* $J(t)$ *at time t is determined by*

$$J(t) = j \quad if \quad U_t^{(j)} > \bigvee_{i\notin j} U_t^{(i)} \qquad i,j \in \mathcal{J}$$

TABLE 1: Assumptions regarding choice behavior and the arrival of stochastic information, and alternative formulations of assumptions about the utility process.

a choice is made.

Of course, a model with an amnesiac treatment of stochastic information can include past information through the inclusion of a measure of lagged choice into the systematic utility, either on theoretical or ad hoc grounds. Using a deterministic parametrization based on a lagged variable that allows for a mean shift in the utility function is a form of "state dependence" will also be considered later in the paper.

It is also of interest to consider which parts of the utility processes are Markovian under the different assumptions. With UP1 a Markovian property holds for $\varepsilon(0,t+\tau] = \varepsilon(0,t] \bigvee \varepsilon(t,t+\tau]$ but not for $\mathbf{U}(0,t+\tau] \neq \mathbf{U}(0,t] \bigvee \mathbf{U}(t,t+\tau]$. With UP2 (including the restriction on μ, a Markovian property holds for $\mathbf{U}'(0,t+\tau] = \mathbf{U}'(0,t] \bigvee \mathbf{U}'(t,t+\tau]$. With UP3 both the total utility $\mathbf{U}^*(0,t+\tau] = \mathbf{U}^*(t,t+\tau] \neq \mathbf{U}^*(0,t] \bigvee \mathbf{U}^*(t,t+\tau]$ and private information $\nu(0,t+\tau] = \nu(t,t+\tau] \neq \nu(0,t] \bigvee \nu(t,t+\tau]$ are non-Markovian.

Following Resnick and Roy, an individual's utility for alternative choices over time can be characterized as a stochastic process of learning about each alternative's utility with an observable or estimable mean. Random deviations from the mean represent private, idiosyncratic information (which are not directly unobserved by a researcher) about an individuals' evaluation of the choices . The arrival of the information about the stochastic utility is random in two ways. The interval between the arrival new information is random, and each new piece of information is of a random value. Assuming information arrives at exponentially distributed intervals and the signal about the utility of a choice is also distributed exponentially, the signals within a segment of time t to $t+\tau$ form a Poisson Random Measure (PRM). The maxima of the signals from this compound process is distributed as a Type I extreme value distribution. Therefore in a segment of length t, the idiosyncratic utility for each choice will be distributed with probability density

$$f_t(v) = te^{-v}e^{-te^{-v}}$$

These properties of the stochastic information are summarized as Assumption SP. The extreme value distribution and independence also follows naturally from the axiomatic approach of Dagsvik (1983).

Regardless of how the total utility is formulated (UP1, UP2 or UP3), the individual choice at each time t is the same. The individual chooses the alternative with the greatest value among the vector of utilities U_t. This is Assumption CP in Table 1. This choice process is also referred to as a discriminal process (Thurstone, 1927) or a leader process (Resnick and Roy, 1990).

4 Choice and Transition Probabilities

The cross-sectional choice probability is well known. An individual choosing between K discrete, unordered categorical alternatives. Using Assumptions SP and CP, along with any one of the UP assumptions (UP1, UP2, or UP2), the probability of choosing category k is the familiar multinomial logit (MNL),

$$\begin{aligned}
\pi_{kt} &:= P(J_t = k) \\
&= P(U_t^{(k)} > U_t^{(1)}, \ldots, U_t^{(k)} > U_t^{(K)}) \\
&= P(\mu_{kt} + \varepsilon_{kt} - \mu_{1t} > \varepsilon_{1t}, \ldots, \mu_{kt} + \varepsilon_{kt} - \mu_{Kt} > \varepsilon_{Kt}) \\
&= \int_{-\infty}^{\infty} \lambda_t(\varepsilon_{kt}) \left[\int_{-\infty}^{\mu_{kt}+\varepsilon_{kt}-\mu_{1t}} \lambda_t(\varepsilon_{1t}) \partial\varepsilon_{1t} \times \cdots \times \int_{-\infty}^{\mu_{kt}+\varepsilon_{kt}-\mu_{Kt}} \lambda_t(\varepsilon_{Kt}) \partial\varepsilon_{Kt} \right] \partial\varepsilon_{kt} \\
&= \int_{-\infty}^{\infty} e^{-\varepsilon_{kt}} \Lambda_t(\varepsilon_{kt})^{1+e^{\mu_{1t}-\mu_{kt}}+\cdots+e^{\mu_{Kt}-\mu_{kt}}} \partial\varepsilon_{kt} \\
&= \frac{e^{\mu_{kt}}}{e^{\mu_{1t}} + \cdots + e^{\mu_{Kt}}}
\end{aligned}$$

Let $K = 2$. Let j be the index of the selected choice, and $\check{j}$ be the index of the other, not chosen alternative. Also let $\Delta_{jt} = \mu_{jt} - \mu_{\check{j}t}$ and $\Delta_{j,t,t+\tau} = \Delta_{j,t+\tau} - \Delta_{jt}$. Therefore $\Delta_{j,t,t+\tau} < 0$ indicates the mean for option j has declined relative to the other option $\check{j}$, similarly $\Delta_{j,t,t+\tau} > 0$ indicates a relative increase, and $\Delta_{j,t,t+\tau} = 0$ indicates no change in the differences. The transition probabilities between categories over time are summarized in the following theorem.

Theorem 1 : *Given the processes defined by Assumptions UP1, SP, and CP for choices $j,k \in \{1,2\}$ at times t and $t+\tau$ ($t,\tau > 0$),*

$$P\left[J_{t+\tau}=j \mid J_t=k\right]=$$

$$\begin{cases} \pi_{j,t+\tau}\dfrac{\tau\pi_{kt}}{t\pi_{k,t+\tau}+\tau\pi_{kt}}+\delta_{kj}\dfrac{t\pi_{k,t+\tau}}{t\pi_{k,t+\tau}+\tau\pi_{kt}} & \textit{if } \Delta_{j,t,t+\tau}\geq 0 \quad (a) \\ \dfrac{\pi_{j,t+\tau}\left[t\left(\pi_{k,t}-\pi_{k,t+\tau}\right)+\tau\left(1-\pi_{k,t}\right)\pi_{k,t}\right]}{\pi_{k,t}\left[t\left(1-\pi_{k,t+\tau}\right)+\tau\left(1-\pi_{k,t}\right)\right]}+\delta_{kj}\dfrac{t\left(1-\pi_{k,t}\right)\pi_{j,t+\tau}}{\pi_{k,t}\left[t\left(1-\pi_{k,t+\tau}\right)+\tau\left(1-\pi_{k,t}\right)\right]} & \textit{if } \Delta_{j,t,t+\tau}\leq 0 \quad (b) \\ \pi_{j,t+\tau}\dfrac{\tau}{t+\tau}+\delta_{kj}\dfrac{t}{t+\tau} & \textit{if } \Delta_{j,t,t+\tau}=0 \quad (c) \end{cases}$$

Proof. See the appendix.

LEMMA 2 : *For transition probabilities defined in Theorem 1, and any $t>0$ and $\tau\geq 0$,*

(i) as the time between the initial and subsequent choice becomes small relative to the time preceding the initial choice,

$$\lim_{\tau/t\to 0} P[J_{t+\tau}=j \mid J_t=k]=\begin{cases} 1 & \textit{if } \Delta_{j,t,t+\tau}\geq 0 \textit{ and } j=k \\ \dfrac{\pi_{j,t+\tau}}{\pi_{j,t}} & \textit{if } \Delta_{j,t,t+\tau}\leq 0 \textit{ and } j=k \\ 0 & \textit{if } \Delta_{j,t,t+\tau}\geq 0 \textit{ and } j\neq k \\ 1-\dfrac{\pi_{k,t+\tau}}{\pi_{k,t}} & \textit{if } \Delta_{j,t,t+\tau}\leq 0 \textit{ and } j\neq k \end{cases}$$

(ii) as the time between the initial and subsequent choice becomes large relative to the time preceding the initial choice,

$$\lim_{\tau/t\to\infty} P[J_{t+\tau}=j \mid J_t=k]=\pi_{j,t+\tau}$$

Proof. See the appendix.

In Theorem 1, the timings of a sequence of observations are identified only as a ratio of the length of times between the events of starting the process or making a choice (τ/t), and therefore the transition probabilities are invariant to multiplicative rescaling of time. Part (i) shows that the probability of remaining with the initial choice is asymmetric as the relative time between the choice τ/t becomes small. For situations where the mean of the earlier choice increases or remains the same, as the amount of time to draw new extrema for the idiosyncratic utility decreases so too does the probability of an individual changing her choice. In contrast, when the mean of the earlier choice decreases relative to the alternative, there is a probability $\pi_{j,t+\tau}/\pi_{j,t}$ of the pre-t idiosyncratic information added to the new means results in a different choice, even if no new stochastic information is received.

COROLLARY 3 : *For transition probabilities defined in Theorem 1,*

$$\text{(i)} \quad \lim_{\tau/t\to 0} P[J_t = k]\cdot P[J_{t+\tau} = j \mid J_t = k] \;=\; \begin{cases} \pi_{j,t} & \text{if } \Delta_{j,t,t+\tau} \geq 0 \text{ and } j = k \\ \pi_{j,t+\tau} & \text{if } \Delta_{j,t,t+\tau} \leq 0 \text{ and } j = k \\ 0 & \text{if } \Delta_{j,t,t+\tau} \geq 0 \text{ and } j \neq k \\ \pi_{k,t} - \pi_{k,t+\tau} & \text{if } \Delta_{j,t,t+\tau} \leq 0 \text{ and } j \neq k \end{cases}$$

$$\text{(ii)} \quad \lim_{\tau/t\to \infty} P[J_t = k]\cdot P[J_{t+\tau} = j \mid J_t = k] \;=\; \pi_{kt}\cdot \pi_{j,t+\tau}$$

Corollary 3 summarizes key implications of the proposed model. Corollary 3(i) notes that as the time between repeated choices becomes small relative to the time to the first choice, the probability of choosing the choice twice repeatedly in quick succession is determined by the lowest probability of choosing j. Specifically, (a) the initial (cross-sectional) choice probability of j if the relative systematic utility of j is the same or increasing over time, or (b) the second-period (cross-sectional) choice probability of j if the relative systematic utility of j is the same or decreasing over time.

Corollary 3(ii) is worthy of particular emphasis. The sequence of choices becomes independent as the time between panels grows large, and therefore the probability of observing a particular sequence of choices is simply the product of the probability of observing the choices at each cross-section. Thus we can also think of the independent repeated choice model not only as the result of having independent stochastic innovations such as in Assumption UP3, but also in the context of the cumulative process in UP1 with relatively large amounts of time between choices.

5 Examples of Heterogeneity in Transition Probabilities

The intuition for why the transition probabilities are asymmetric depending on the direction of the changes over time in the systematic utilities is as follows. Consider a situation of an individual who initially chose a category with a very low relative systematic utility component (significantly lower than the alternative choice). Having chosen this category, we know something about the lower bound of the differences between the unobserved stochastic utilities for this individual: it must be very large. If at the next time the individual makes a decision the systematic component has increased in relative size compared to the alternatives, we have a good idea of how likely it is that the individual will make a different choice: combining the increased systematic mean with the low probability that another category has revealed a stochastic utility value larger than the unusual value seen for the initial choice, it is very unlikely that the individual will make a different choice. Consider the opposite scenario, where the individual initially chose the category with the high systematic mean: this is neither surprising nor particularly informative about the stochastic components up to that point. If the systematic mean decreases for that choice, the history of the stochastic process does not increase the probability of remaining in that category.

Specific numerical examples may also help make to make these ideas concrete. In particular, consider individuals who made the same choice in the first period, but with different systematic utility values. Table 2 shows the results for different μ and different τ. Consider first the upper panel of Table 2, with $t = 10$ and $\tau = 10$. In all cases, choice 2 has the same systematic utility $\mu_{2t} = \mu_{2,t+\tau} = 1$, and only the systematic utility of choice 1 varies. Person C is the base case, with the systematic utilities equal ($\mu_{1t} = \mu_{2t}$) and no change in the second period. Having chosen category 1 in the first period, the probability of choosing this category again is 0.75. Given equality of the systematic utilities, the independence over time of the stochastic utilities, and $t = \tau$, one could verify this probability in a straightforward manner.[3]

For persons A and B, who have a low initial probability of choosing category 1 ($P_t = 0.27$), the increase in μ_1 makes choosing the same category in period two even more likely, .99 and .83 respectively. The greater

[3] For two independent processes during two non-overlapping periods, the largest value is equally likely to occur during either period. Therefore, there is a 1/2 chance that the biggest utility occurred in the first period, where we know by conditioning that choice 1 was made; this contributes probability $1/2 \times 1$ to picking 1 again in period two. In the second period, each choice is equally likely to see a (non-cumulative) extreme; multiplying the chance that it was choice 1 and the chance that a new extreme was observed overall in period two, we have $1/2 \times 1/2$. The sum of the probability of these two possible outcomes is the probability of remaining with choice 1 in period two given a choice of 1 in period one. I.e., there is a $0.5 + 0.5 \times 0.5 = 0.75$ probability of the cumulative maximum of the stochastic utility of choice 1 being greater than the cumulative maximum for choice 2.

Time intervals $t = 10$ and $\tau = 10$:

Obs	μ_2	μ_{1t}	$\mu_{1,t+\tau}$	$(\Delta_{t+\tau} - \Delta_t)$	$P_{1,t}$	$P_{1,t+\tau}$	$P[J_{t+\tau} = 1 \mid J_t = 1]$
A	1	0	4	Inc.	0.27	0.95	0.99
B	1	0	1	Inc.	0.27	0.50	0.83
C	1	1	1	0	0.50	0.50	0.75
D	1	2	1	Dec.	0.73	0.50	0.62
E	1	4	0	Dec.	0.95	0.27	0.28

Time intervals $t = 10$ and $\tau = 200$:

Obs	μ_2	μ_{1t}	$\mu_{1,t+\tau}$	$(\Delta_{t+\tau} - \Delta_t)$	$P_{1,t}$	$P_{1,t+\tau}$	$P[J_{t+\tau} = 1 \mid J_t = 1]$
A	1	0	4	Inc.	0.27	0.95	0.96
B	1	0	1	Inc.	0.27	0.50	0.54
C	1	1	1	0	0.50	0.50	0.52
D	1	2	1	Dec.	0.73	0.50	0.52
E	1	4	0	Dec.	0.95	0.27	0.27

TABLE 2: Examples of transition probabilities from the dynamic MNL model

increase makes leaving the category all the less likely, as fits with general intuition. For person D, who was quite likely to have chosen category 1 initially, the decrease in μ_1 means that the probability is not as high as it was for person C who did not change in systematic utilities, but still greater than $P_{t+\tau}(1)$. Person E–who experienced the largest decrease in choice 1 utility has a probability of remaining with the choice approaching the unconditional probability of choosing category 1.

One approach to estimating transition matrices is to condition on the subset of individuals who made the same choice at period t, and estimate a MNL for the probability of choosing a category at period $t + \tau$. This method pools together those individuals with increasing and decreasing systematic utilities with the result that the estimated transition probabilities will depend entirely on the distribution of the individuals with different utilities.

For example, consider the people in Table 2 and the results in the upper panel. If the entire sample were made up of individuals like person B, then the conditional MNL approach would produce the correct transition probabilities, as shown in Case 1 of Table 3. Similarly, the correct results would be obtained if the sample were made up entirely of people like persons D (Case 2).

Because this conditional method estimates the average utilities given the observed past choice, people

Case 1. All Person A – correct probabilities:

Choice in period 1		Choice in period 2: 1	2
	1	0.99	0.01
	2	0.94	0.06

Case 2. All Person E – correct probabilities:

Choice in period 1		Choice in period 2: 1	2
	1	0.28	0.72
	2	0.02	0.98

Case 3. Equal mixture of A and E – unrepresentative probabilities;

Choice in period 1		Choice in period 2: 1	2
	1	0.64	0.36
	2	0.48	0.52

TABLE 3: Examples of transition matrices from conditional MNL, using cases described in Table 2 with $t = 10$ and $\tau = 10$.

of type B and D may be treated as identical in Case 3 even though they differ in first period utilities. By mixing different proportions of these people together, some with increasing and others with decreasing utilities, it is possible to get an arbitrary transition matrix which is a weighted average of these two groups, but which is not representative of either. The conditional MNL method will produce the correct average probabilities *but will only be accurate characterizations of individual behavior if everyone in the sample has the same vector of systematic utilities in both periods*. Otherwise the results will be unreprentative of any individual and thus of questionable use for making inferences about the systematic utilities.

Another method is to estimate a cross-sectional MNL model but with dummies representing the category of choice from the previous period. Denote this model the MNL with lagged dependent variable (LDV). This essentially restricts changes between $y = 1$ and $y = 0$ to be summarized by a mean shift in the linear aggregator function, thereby simply moving the logistic ogive left (right) if the coefficient is positive (negative) and the lagged value $y = 1$.

A common justification for this is to account for a notion of state or path dependence. However, this method can also be very misleading. The intuition of the problem with the method is that one can get spuriously significant effects on the lagged values simply because of the correlation between the cumulative maxima of the stochastic utilities over time. The proportion of cases with increasing and decreasing systematic utilities can also induce arbitrary coefficients on these lagged coefficients.

Consider again the hypothetical people from Table 2, beginning with person C, who did not change utilities and has the same marginal probability (0.5) of choosing category 1 in both periods. If a sample were composed entirely of Person C then the dummy variable would pickup the difference between the marginal probability of choosing category 1 and the conditional probability $P[J_{t+\tau} = j \mid J_t = j] = .75$, i.e., the LDV would attribute an increase of 0.25 in the probability of choosing category 1 to "state dependence".

As an example of the effect of changing systematic utilities, consider an even mixture between persons of type B and of type D. Both these groups have marginal probabilities of 0.5 for choosing choice 1 in the second period ($t+\tau$). However, the conditional probabilities ($P[J_{t+\tau} = j \mid J_t = j]$) are .83 and .62 respectively. The LDV model will have the dummy pickup the difference between the average of these conditional probabilities and the marginal value, .725 - .5 = .125.

6 Conclusion

The theory developed in this paper investigates the transition probabilities between choices for individuals, where past idiosyncratic preferences are not ignored each time a choice is made. Though they are not forgotten, they can be superseded by new information. In this paper I have adopted a particular formulation of this process and used it to derived closed form solutions to the transition probabilities. The main theorem, and it's lemma help to illuminate how choice behavior is affected when individuals are observed at snapshots of an evolving cumulative learning process. Additional examples and further results on evaluating the empirical appropriateness of the model remain to be included.

7 Appendix I

Proof of Theorem 1: For the results that follow, I use Assumptions UP2, SP, and CP. Part (a) can be proved by decomposing the integral into components which represent the two cases where the utility of the observed choice j exceeds that of the unchosen alternative $\check{j}$ in period two. Defining $\phi_{jt} = \varepsilon_{\check{j}t} - \varepsilon_{jt}$, the probability of repeating the same choice j given $\Delta_{jt} < \Delta_{jt+\tau}$ is,

$$
\begin{aligned}
&P[J(t+\tau) = j \mid J(t) = j, \Delta_{jt} < \Delta_{jt+\tau}] \\
&= P(\phi_{jt+\tau} < \Delta_{jt+\tau} \mid \phi_{jt} < \Delta_{jt}, \Delta_{jt} < \Delta_{jt+\tau}) \\
&= P(\phi_{jt+\tau} < \Delta_{jt+\tau}, \phi_{jt} < \Delta_{jt} \mid \Delta_{jt} < \Delta_{jt+\tau}) / P(\phi_{jt} < \Delta_{jt}) \\
&= \frac{1}{\pi_{jt}} \int_{-\infty}^{\infty} \lambda_t(\varepsilon_{jt}) \int_{-\infty}^{\varepsilon_{jt}+\Delta_t} \lambda_t(\varepsilon_{\check{j}t}) \partial \varepsilon_{\check{j}t} \times \\
&\qquad \left[\int_{-\infty}^{\varepsilon_{jt}+\Delta_{jt+\tau}} \lambda_\tau(\varepsilon_{\check{j}t+\tau}) \partial \varepsilon_{\check{j}t+\tau} + \int_{\varepsilon_{jt}+\Delta_{jt+\tau}}^{\infty} \lambda_\tau(\varepsilon_{\check{j}t+\tau}) \int_{\varepsilon_{\check{j}t+\tau}-\Delta_{jt+\tau}}^{\infty} \lambda_\tau(\varepsilon_{jt+\tau}) \partial \varepsilon_{jt+\tau} \partial \varepsilon_{\check{j}t+\tau} \right] \partial \varepsilon_{jt} \\
&= \frac{1}{\pi_{jt}} \int_{-\infty}^{\infty} \lambda_t(\varepsilon_{jt}) \Lambda_t(\varepsilon_{jt} + \Delta_{jt}) \times \\
&\qquad \left[\Lambda_\tau(\varepsilon_{jt} + \Delta_{jt+\tau}) + \int_{\varepsilon_{jt}+\Delta_{jt+\tau}}^{\infty} \lambda_\tau(\varepsilon_{\check{j}t+\tau})(1 - \Lambda_\tau(\varepsilon_{\check{j}t} - \Delta_{jt+\tau})) \partial \varepsilon_{\check{j}t+\tau} \right] \partial \varepsilon_{jt} \\
&= \frac{1}{\pi_{jt}} \int_{-\infty}^{\infty} \lambda_t(\varepsilon_{jt}) \Lambda_t(\varepsilon_{jt} + \Delta_{jt}) \times \\
&\qquad [\Lambda_\tau(\varepsilon_{jt} + \Delta_{jt+\tau}) + 1 - \Lambda_\tau(\varepsilon_{jt} + \Delta_{jt+\tau}) - (1 - \pi_{jt+\tau})(1 - \Lambda_\tau(\varepsilon_{jt} + \Delta_{jt+\tau}) \Lambda_\tau(\varepsilon_{jt}))] \partial \varepsilon_{jt} \\
&= \frac{1}{\pi_{jt}} \int_{-\infty}^{\infty} \lambda_t(\varepsilon_{jt}) \Lambda_t(\varepsilon_{jt} + \Delta_{jt}) [\pi_{jt+\tau} + (1 - \pi_{jt+\tau}) \Lambda_\tau(\varepsilon_{jt} + \Delta_{jt+\tau}) \Lambda_\tau(\varepsilon_{jt}))] \partial \varepsilon_{jt} \\
&= \frac{\pi_{jt+\tau}}{\pi_{jt}} \frac{1}{1 + e^{-\Delta_{jt}}} + \frac{(1 - \pi_{jt+\tau})}{\pi_{jt}} \frac{t}{t + te^{-\Delta_{jt}} + \tau e^{-\Delta_{jt+\tau}} + \tau} \\
&= \pi_{jt+\tau} \frac{\tau \pi_{jt}}{\tau \pi_{jt} + t \pi_{jt+\tau}} + \frac{t \pi_{jt+\tau}}{\tau \pi_{jt} + t \pi_{jt+\tau}}
\end{aligned}
$$

and more generally for any $j,k \in \{1,2\}$ the transition probability is, probability is,

$$
\begin{aligned}
& P[J(t+\tau) = j \mid J(t) = k, \Delta_{jt} < \Delta_{jt+\tau}] \\
& \quad = \quad P(\phi_{jt+\tau} < \Delta_{jt+\tau} \mid \phi_{kt} > \Delta_{kt}, \Delta_{jt} < \Delta_{jt+\tau}) \\
& \quad = \quad P(\phi_{jt+\tau} < \Delta_{jt+\tau}, \phi_{kt} > \Delta_{kt} \mid \Delta_{jt} < \Delta_{jt+\tau}) / P(\phi_{kt} < \Delta_{kt}) \\
& \quad = \quad \pi_{j,t+\tau} \frac{\tau \pi_{kt}}{t \pi_{k,t+\tau} + \tau \pi_{kt}} + \delta_{kj} \frac{t \pi_{k,t+\tau}}{t \pi_{k,t+\tau} + \tau \pi_{kt}}
\end{aligned}
$$

where $\delta_{kj} = 1$ if $k = j$, and zero otherwise.

For part (b), the probability of repeating the same choice j given $\Delta_{jt} > \Delta_{jt+\tau}$ is,

$$
\begin{aligned}
& P[J(t+\tau) = j \mid J(t) = j, \Delta_{jt} > \Delta_{jt+\tau}] \\
& = P(\phi_{jt+\tau} < \Delta_{jt+\tau} \mid \phi_{jt} < \Delta_{jt}, \Delta_{jt} > \Delta_{jt+\tau}) \\
& = P(\phi_{jt+\tau} < \Delta_{jt+\tau}, \phi_{jt} < \Delta_{jt} \mid \Delta_{jt} > \Delta_{jt+\tau}) / P(\phi_{jt} < \Delta_{jt}) \\
& = \frac{1}{\pi_{jt}} \int_{-\infty}^{\infty} \lambda_t(\varepsilon_{jt}) \int_{-\infty}^{\varepsilon_{jt} + \Delta_{jt+\tau}} \lambda_t(\varepsilon_{\check{j}t}) \partial \varepsilon_{\check{j}t} \left[\int_{-\infty}^{\varepsilon_{jt} + \Delta_{jt+\tau}} \lambda_\tau(\varepsilon_{\check{j}t+\tau}) \partial \varepsilon_{\check{j}t+\tau} + \right. \\
& \qquad \left. \int_{\varepsilon_{jt} + \Delta_{jt+\tau}}^{\infty} \lambda_\tau(\varepsilon_{\check{j}t+\tau}) \int_{\varepsilon_{\check{j}t+\tau} - \Delta_{jt+\tau}}^{\infty} \lambda_\tau(\varepsilon_{jt+\tau}) \partial \varepsilon_{jt+\tau} \partial \varepsilon_{\check{j}t+\tau} \right] \partial \varepsilon_{jt} + \\
& \quad \frac{1}{\pi_{jt}} \int_{-\infty}^{\infty} \lambda_t(\varepsilon_{jt}) \int_{\varepsilon_{jt} + \Delta_{jt+\tau}}^{\varepsilon_{jt} + \Delta_{jt}} \lambda_t(\varepsilon_{\check{j}t}) \partial \varepsilon_{\check{j}t} \left[\int_{-\infty}^{\varepsilon_{\check{j}t}} \lambda_\tau(\varepsilon_{\check{j}t+\tau}) \int_{\varepsilon_{\check{j}t} - \Delta_{jt+\tau}}^{\infty} \lambda_\tau(\varepsilon_{jt+\tau}) \partial \varepsilon_{jt+\tau} \partial \varepsilon_{\check{j}t+\tau} + \right. \\
& \qquad \left. \int_{\varepsilon_{\check{j}t}}^{\infty} \lambda_\tau(\varepsilon_{\check{j}t+\tau}) \int_{\varepsilon_{\check{j}t+\tau} - \Delta_{jt+\tau}}^{\infty} \lambda_\tau(\varepsilon_{jt+\tau}) \partial \varepsilon_{jt+\tau} \partial \varepsilon_{\check{j}t+\tau} \right] \partial \varepsilon_{jt} \\
& = \frac{\pi_{jt+\tau}}{\pi_{jt}} \left[\pi_{jt+\tau} + (1 - \pi_{jt+\tau}) \frac{t}{t+\tau} \right] + \\
& \quad \frac{\pi_{jt+\tau}}{\pi_{jt}} \left[\pi_{jt} - \pi_{jt+\tau} - \frac{t(1 - \pi_{jt+\tau})}{t(1 - \pi_{jt+\tau}) + \tau} \left\{ \frac{t \pi_{jt}(1 - \pi_{jt+\tau})}{t(1 - \pi_{jt+\tau}) + \tau(1 - \pi_{jt})} - \frac{t \pi_{jt+\tau} \pi_{jt+\tau}}{t \pi_{jt+\tau} + \tau(1 - \pi_{jt+\tau})} \right\} \right] \\
& = \frac{\pi_{j,t+\tau} \left[t\,(\pi_{j,t} - \pi_{j,t+\tau}) + \tau\,(1 - \pi_{j,t}) \pi_{j,t} \right]}{\pi_{j,t} \left[t\,(1 - \pi_{j,t+\tau}) + \tau\,(1 - \pi_{j,t}) \right]} + \frac{t\,(1 - \pi_{j,t})\,\pi_{j,t+\tau}}{\pi_{j,t} \left[t\,(1 - \pi_{j,t+\tau}) + \tau\,(1 - \pi_{j,t}) \right]}
\end{aligned}
$$

and for general $j,k \in \{1,2\}$,

$$
\begin{aligned}
&P\left[J(t+\tau) = j \mid J(t) = k, \Delta_{jt} > \Delta_{jt+\tau}\right] \\
&= \frac{\pi_{j,t+\tau}\Big[t\,(\pi_{k,t} - \pi_{k,t+\tau}) + \tau\,(1-\pi_{k,t})\pi_{k,t}\Big]}{\pi_{k,t}\Big[t\,(1-\pi_{k,t+\tau}) + \tau\,(1-\pi_{k,t})\Big]} + \delta_{kj}\frac{t\,(1-\pi_{k,t})\,\pi_{j,t+\tau}}{\pi_{k,t}\Big[t\,(1-\pi_{k,t+\tau}) + \tau\,(1-\pi_{k,t})\Big]}
\end{aligned}
$$

Part (c) follows (a) and (b), as $\Delta_{jt} - \Delta_{jt+\tau} \to 0$; see also Corollary 3.5 of Resnick and Roy (1990).

PROOF OF LEMMA 2: For transition probabilities defined in Theorem 1, and any $t > 0$ and $\tau \geq 0$, note

$$
\begin{aligned}
P[J_{t+\tau} = j \mid J_t = k\ ,\ \Delta_{j,t,t+\tau} \geq 0] &= \pi_{j,t+\tau}\frac{(\tau/t)\pi_{kt}}{\pi_{k,t+\tau} + (\tau/t)\pi_{kt}} + \delta_{kj}\frac{\pi_{k,t+\tau}}{\pi_{k,t+\tau} + (\tau/t)\pi_{kt}} \\
&= \pi_{j,t+\tau}\frac{\pi_{kt}}{(t/\tau)\pi_{k,t+\tau} + \pi_{kt}} + \delta_{kj}\frac{(t/\tau)\pi_{k,t+\tau}}{(t/\tau)\pi_{k,t+\tau} + \pi_{kt}}
\end{aligned}
$$

such that

$$
\begin{aligned}
\lim_{\tau/t \to 0} P[J_{t+\tau} = j \mid J_t = k\ ,\ \Delta_{j,t,t+\tau} \geq 0\,] &= \pi_{j,t+\tau}\frac{0}{\pi_{k,t+\tau}} + \delta_{kj}\frac{\pi_{k,t+\tau}}{\pi_{k,t+\tau}} = \delta_{kj} \\
\lim_{t/\tau \to 0} P[J_{t+\tau} = j \mid J_t = k\ ,\ \Delta_{j,t,t+\tau} \geq 0\,] &= \pi_{j,t+\tau}\frac{\pi_{kt}}{\pi_{kt}} + \delta_{kj}\frac{0}{\pi_{kt}} = \pi_{j,t+\tau}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
P[J_{t+\tau} = j \mid J_t = k\ ,\ \Delta_{j,t,t+\tau} \leq 0] &= \frac{\pi_{j,t+\tau}\Big[\,(\pi_{k,t} - \pi_{k,t+\tau}) + (\tau/t)\,(1-\pi_{k,t})\pi_{k,t}\Big]}{\pi_{k,t}\Big[\,(1-\pi_{k,t+\tau}) + (\tau/t)\,(1-\pi_{k,t})\Big]} + \delta_{kj}\frac{(1-\pi_{k,t})\,\pi_{j,t+\tau}}{\pi_{k,t}\Big[\,(1-\pi_{k,t+\tau}) + (\tau/t)\,(1-\pi_{k,t})\Big]} \\
&= \frac{\pi_{j,t+\tau}\Big[(t/\tau)\,(\pi_{k,t} - \pi_{k,t+\tau}) + \,(1-\pi_{k,t})\pi_{k,t}\Big]}{\pi_{k,t}\Big[(t/\tau)\,(1-\pi_{k,t+\tau}) + \,(1-\pi_{k,t})\Big]} + \delta_{kj}\frac{(t/\tau)\,(1-\pi_{k,t})\,\pi_{j,t+\tau}}{\pi_{k,t}\Big[(t/\tau)\,(1-\pi_{k,t+\tau}) + \,(1-\pi_{k,t})\Big]}
\end{aligned}
$$

such that,

$$\begin{aligned}
\lim_{\tau/t\to 0} P[J_{t+\tau}=j \mid J_t=k\ ,\ \Delta_{j,t,t+\tau}\le 0] &= \frac{\pi_{j,t+\tau}\,(\pi_{k,t}-\pi_{k,t+\tau})+\delta_{kj}\,(1-\pi_{k,t})\,\pi_{j,t+\tau}}{\pi_{k,t}\,(1-\pi_{k,t+\tau})} \\
&= (1-\delta_{kj})+\frac{\pi_{k,t+\tau}}{\pi_{k,t}}(-1)^{(1-\delta_{kj})} \\
\lim_{t/\tau\to 0} P[J_{t+\tau}=j \mid J_t=k\ ,\ \Delta_{j,t,t+\tau}\le 0] &= \frac{\pi_{j,t+\tau}\,(1-\pi_{k,t})\pi_{k,t}}{\pi_{k,t}\,(1-\pi_{k,t})}+\delta_{kj}\frac{0}{\pi_{k,t}\,(1-\pi_{k,t})}=\pi_{j,t+\tau}
\end{aligned}$$

APPENDIX II: ALTERNATIVE MODELS

Resnick and Roy (1990) show that if the systematic utility component of each choice does not change, then the transition probabilities are characterized by the function,

$$P\left[J_t=j \mid J_\tau=i\right]=\begin{cases}\pi_{jt}-\rho_{\tau,t}\pi_{j\tau} & \text{if } j\neq i \\ \\ \pi_{jt}-\rho_{\tau,t}\pi_{j\tau}+\rho_{\tau,t} & \text{if } j=i\end{cases} \tag{1}$$

where π_{jt} is the (cross-sectional) probability of choosing category j at time t, and the serial correlation between choices at times t and $t+\tau$ is a function of the correlation between the utilities of the chosen categories,

$$\rho_{\tau,t}=\text{Corr}\left(\frac{1}{Z_t},\frac{1}{Z_{t+\tau}}\right)$$

Note that J_t and Z_t for $t>0$ are both Markov.

If the errors are distributed as Type I extreme values then the probability terms are simply the multinomial logit values,

$$\pi_{jt}=\frac{e^{\mu_{jt}}}{\sum e^{\mu_{it}}}$$

with transition probabilities,

$$P\left[J_t=j \mid J_\tau=i\right]=\pi_{j,t+\tau}\frac{\tau}{t+\tau}+\delta_{ij}\frac{t}{t+\tau} \tag{2}$$

Unlike the ad hoc or conditional methods for estimating transition probabilities, this method is a generalization of the MNL. As $\tau \to 0$, the probability of remaining in the same category goes to 1. It is possible to further generalize this model by allowing for correlations between the stochastic utilities of the choices, resulting in models such as a dynamic GEV.

Roy, Chintagunta, and Haldar (1996) provide empirical analyis of repeated consumer purchases which draws on the model of Resnick and Roy (1990) to model a process with time-varying utilities as a Markov chain. Indeed Roy, Chintagunta, and Haldar (1996) is the only published empirical application of the Resnick and Roy technique. Their specification is,

$$
\begin{aligned}
\Pr\left(J_{t+\tau} = j \mid J_t = l\right) &= \left(1-\tilde{\rho}_{t-\tau}\right)\pi_{jt} + \tilde{\rho}_{t-\tau}\delta_{it}(j=l) \\
\tilde{\rho}_{t-\tau} &= \exp\left\{-\tilde{\gamma}(t-\tau)\right\} \qquad \tilde{\gamma} > 0
\end{aligned}
$$

This will be referred to as the RCH model. This model makes a number of strong assumptions.

RR V RCH: The relationship between Resnick and Roy (1990, RR) Corollary 3.5.b and the model suggested by Roy, Chintagunta, and Haldar (1996, RCH) is characterized by the restrictions necessary for their transition probabilities to be equal:

$$
\begin{aligned}
\pi_{jt} - \pi_{j\tau}\frac{\sum_{k\in\mathcal{P}}\exp\left\{v_\tau^{(k)}\right\}}{\sum_{k\in\mathcal{P}}\exp\left\{v_t^{(k)}\right\}}\rho_{t-\tau} + \frac{\sum_{k\in\mathcal{P}}\exp\left\{v_\tau^{(k)}\right\}}{\sum_{k\in\mathcal{P}}\exp\left\{v_t^{(k)}\right\}}\rho_{t-\tau}\delta(j=l) &= \pi_{jt} - \pi_{jt}\tilde{\rho}_{t-\tau} + \tilde{\rho}_{t-\tau}\delta(j=l) \\
-\pi_{jt}\frac{\exp\left\{v_\tau^{(j)}\right\}}{\exp\left\{v_t^{(j)}\right\}}\rho_{t-\tau} + \frac{\sum_{k\in\mathcal{P}}\exp\left\{v_\tau^{(k)}\right\}}{\sum_{k\in\mathcal{P}}\exp\left\{v_t^{(k)}\right\}}\rho_{t-\tau}\delta(j=l) &= -\pi_{jt}\tilde{\rho}_{t-\tau} + \tilde{\rho}_{t-\tau}\delta(j=l)
\end{aligned}
$$

so the models are equivalent when the following condition holds,

$$
\frac{\sum_{k\in\mathcal{P}}\exp\left\{v_\tau^{(k)}\right\}}{\sum_{k\in\mathcal{P}}\exp\left\{v_t^{(k)}\right\}}\rho_{t-\tau} = \frac{\exp\left\{v_\tau^{(j)}\right\}}{\exp\left\{v_t^{(j)}\right\}}\rho_{t-\tau} = \tilde{\rho}_{t-\tau}
$$

These restrictions hold when $\sum_{k\in\mathcal{P}}\exp\left\{v_\tau^{(k)}\right\} = \sum_{k\in\mathcal{P}}\exp\left\{v_t^{(k)}\right\}$ and $\exp\left\{v_\tau^{(j)}\right\} = \exp\left\{v_t^{(j)}\right\}$ such that the state probabilities are homogeneous across time, $\pi_{jt} = \pi_{j\tau}$, and the serial autocorrelation is the same $\rho_{t-\tau} = \tilde{\rho}_{t-\tau}$. This is an untenable a priori assumption if a theory hypothesizes changes in individual utility calculations over time.

Slightly less implausible is that the utilities vary while the ratio of the measure of choice j at times t and τ is equivalent to the ratio of the total measures. Since $\rho_{t-\tau}$ is not necessarily constant, the ratio need not be constant. This restriction could be achieved if the measure of every choice grew at the same (although potentially time-varying) rate, proportional to its size at time τ.

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