Statistical Methods III: Spring 2013

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B-spline + constrained inference

Outline

Basis splines

Logic:

- map $x_i \rightarrow h_m(x_i)$ (x_i into M basis functions)
- estimate $f(x)$ (curve, a weighted sum of $h_m(x)$):

$$\hat{f}(x_i) = \sum_{m=1}^{M} \hat{\beta}_m h_m(x_i) = \beta h(x_i)$$

where $\hat{\beta} = \left[\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_m\right]$ are simply regression coefficients.

Features of B-splines

- shapes can be described by linear functions of β
- $h_m(x)$ has local support, β_m has local effect

Basis splines, basic logic

- Given a choice of knot locations $\{\lambda_1, ..., \lambda_M\}$ and polynomial order k
- Decompose x_i into $M + 2$ basis functions, with mth

$$h_{m,k+1}(x) = (\lambda_{m+k+1} - \lambda_i) \sum_{j=0}^{k+1} \frac{(\lambda_{m+j} - x)_+^k}{\prod_{l=0,l\neq j}^{k+1}(\lambda_{m+j} - \lambda_{m+l})}$$

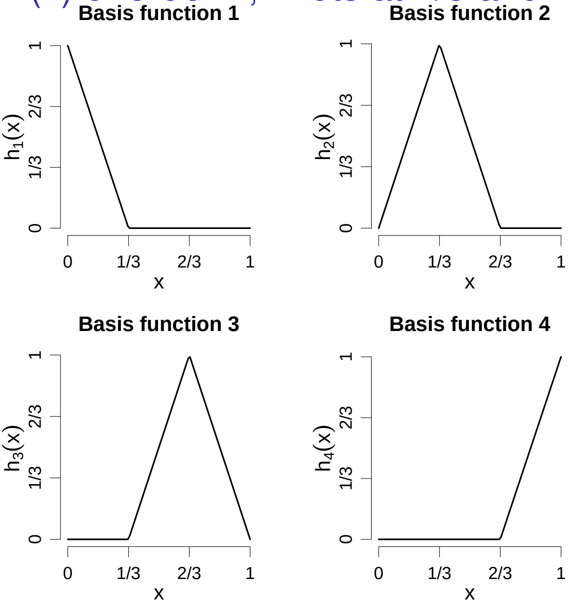
- for a vector of spline coefficients,

$$\hat{\beta} = \left[\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_m\right]$$

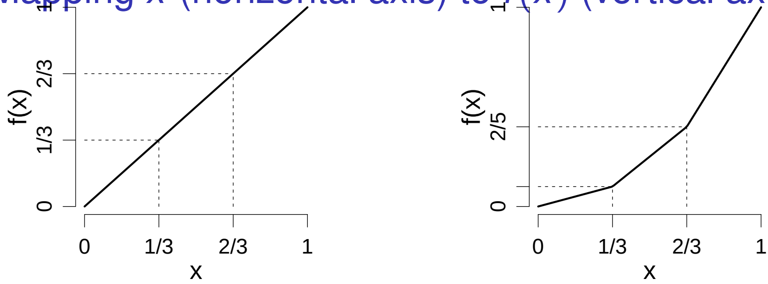
- $f(x)$ is a weighted sum of $h_m(x)$

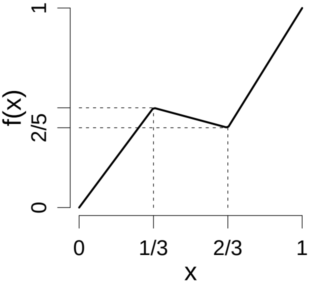
$$\hat{f}(x_i) = \sum_{m=1}^{M} \hat{\beta}_m h_m(x_i)$$

A look at h(x) of order 1, knots at 1/3 and 2/3



Mapping x (horizontal axis) to $f(x)$ (vertical axis)





$$\begin{aligned}
\text{(a)} \quad \theta &= [\,0,\ .33,\ .66,\ 1] \\
\Delta\theta &= [\,.33,\ .33,\ .33] \\
\text{(b)} \quad \theta &= [\,0,\ .1,\ .4,\ 1] \\
\Delta\theta &= [\,1,\ .3,\ .6] \\
\text{(c)} \quad \theta &= [\,0,\ .5,\ .4,\ 1] \\
\Delta\theta &= [\,.5,\ -.1,\ .6]
\end{aligned}$$

Linear constraints implying shapes

	Restriction	$f(x)$	*Interval*
1)	$\beta_m - \beta_{m-1} > 0$	increasing	(k_{m-1}, k_m)
2)	$\beta_m - \beta_{m-1} = 0$	flat	(k_{m-1}, k_m)
3)	$\beta_m - \beta_{m-1} < 0$	decreasing	(k_{m-1}, k_m)
4)	$\frac{\beta_{m+1} - \beta_m}{k_{m+1} - k_m} = \frac{\beta_m - \beta_{m-1}}{k_m - k_{m-1}}$	linear	(k_{m-1}, k_{m+1})
5)	$\frac{\beta_{m+1} - \beta_m}{k_{m+1} - k_m} > \frac{\beta_m - \beta_{m-1}}{k_m - k_{m-1}}$	convex	(k_{m-1}, k_{m+1})
6)	$\frac{\beta_{m+1} - \beta_m}{k_{m+1} - k_m} < \frac{\beta_m - \beta_{m-1}}{k_m - k_{m-1}}$	concave	(k_{m-1}, k_{m+1})

and can combine, e.g., monotonic and convex; unimodal

Linear constraints implying B-spline shapes

Linear restrictions on parameters, can be written as,

$$R\beta - c \geq 0$$

Example 1: monotonicity ($\beta_m - \beta_{m-1} > 0$)

$$R_m = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \; \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}, \; c = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 2: symmetric

$$R_m = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \; \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}, c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Estimation of spline coefficients

- Unconstrained shape, $f(x)$ is linear function of β

$$\min \sum_{i}^{N} (y_i - h(x_i)\hat{\beta})^2$$

- Constrained OLS: quadratic programing problem

$$\min \sum_{i}^{N} (y_i - h(x_i)\tilde{\beta})^2 \quad \text{subject to } R\tilde{\beta} - c \geq 0$$

- Constrained, non-linear/ML: logarithmic barrier

$$\sum_{i}^{N} L(h(x_i)\tilde{\beta}) - \mu \sum \log(R\tilde{\beta} - c)$$